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THE ROLLING OF A WHEEL WITH A REINFORCED TYRE ALONG A PLANE WITHOUT SLIP†

V. G. VIL'KE and I. F. KOZHEVNIKOV

Moscow

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A model of a wheel with a reinforced tyre, whose surface is simulated by a flexible strip (tread) attached to parts of two tori (the sidewalls of the tyre) is considered. The disk of the wheel (a rigid body) has six degrees of freedom and is in contact with the plane along part of the tread. Based on several assumptions, the potential energy functional of the deformed wheel is found as a function of the deformations of the centre line of the tread. On the assumption that the wheel is rolling without slip in the region of contact of the tread with the plane along a previously unknown section of the tread, the complete system of equations of motion is obtained. The equilibrium of the wheel and the steady state of rolling in a straight line with given swivel and tilt are investigated, and all characteristics of the motion are found (the contact region, the tyre deformation, and the forces and torques applied to the wheel disk). © 2002 Elsevier Science Ltd. All rights reserved.

There are several well-known models for a tyre whose deformations are described by a finite number of parameters, such as the displacement and rotation of the load surface [1–4]. The dynamic effects due to deformation of the tyre over its entire surface can be described in the context of models with an infinite number of degrees of freedom [5, 6]. Metelitsyn [7] has suggested modelling the tyre surface by part of the surface of a torus, but then goes on to reduce the deformations to displacement of the load curve along the wheel axis, taking the force and torque to be proportional to this displacement and its derivative with respect to natural parameter at the contact point. Böhm [5] proposes modelling a pneumatic tyre by a curved beam attached to a disk by continuously distributed elastic forces. Tyres have also been simulated [6] by a tread (a flexible inextensible thread) attached to sidewalls (parts of the surface of a torus), with the assumption that the middle plane of the wheel disk is orthogonal to the plane of rolling. There are a good many publications investigating tyre deformations by the finite-element method (e.g., [8, 9]). Unlike those publications, in this paper we propose a wheel model in which the disk has six degrees of freedom, the tread is represented by a flexible inextensible strip, and the sidewalls are simulated by parts of the surfaces of two tori; this model enables the rolling of the wheel to be investigated most completely.

1. THE MODEL OF A WHEEL WITH A REINFORCED TYRE

Let the wheel consist of a disk (0), two tyre sidewalls (1, 2), and a tread (3), represented in the undeformed state by a cylindrical surface of radius r. The wheel disk will be treated as a rigid body, whose position is determined by six degrees of freedom, and the tyre sidewalls in the undeformed state will be defined as two parts of toroidal surfaces (Fig. 1). Let $OX_1X_2X_3$ be an inertial system of coordinates and let $Cxyz$ be a system of coordinates attached to the disk. The tread surface is defined by

$$\mathbf{R}_3(\varphi,\xi,t)=\sum_{i=1}^{3}X_i\mathbf{I}_i+r\Gamma_3(\beta)\Gamma_1(\varkappa)\Gamma_2(\theta+\varphi)\left[(1+U_1)\mathbf{e}_1+\left(lr^{-1}\xi+U_2\right)\mathbf{e}_2+U_3\mathbf{e}_3\right] \tag{1.1}$$

$$\Gamma_1(\varkappa)=\begin{Vmatrix}1 & 0 & 0\\ 0 & \cos\varkappa & -\sin\varkappa\\ 0 & \sin\varkappa & \cos\varkappa\end{Vmatrix},\quad \Gamma_2(\theta)=\begin{Vmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{Vmatrix}$$

$$\Gamma_3(\beta)=\begin{Vmatrix}\cos\beta & -\sin\beta & 0\\ \sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{Vmatrix},\quad \varphi \bmod 2\pi,\quad |\xi|\le 1$$

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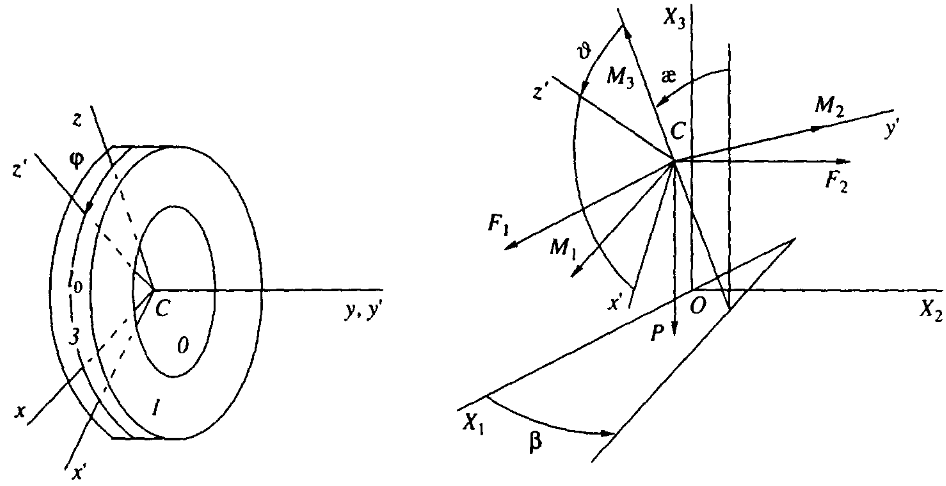

Fig. 1

where X_1, X_2 and X_3 are the coordinates of the mass centre of the disk C, β, $\varkappa$ and θ are the angles of successive rotations of the fixed system of coordinates about the corresponding axes on transferring to the system $Cxyz$, $\mathbf{I}_i$ is a unit vector on the axis OX_i, $U_k(\xi, \varphi, t)$ $(k = 1, 2, 3)$ are the components of the displacement vector of a point of the tread in a system of coordinates $Cx'y'z'$ whose unit vectors are denoted by $\mathbf{e}_k$, and $2l$ is the tread width. We will assume that the tyre is reinforced in the zone of the tread so that a fibre passes through each point in the direction of $\mathbf{e}_2$ and two fibres in the directions $\cos l\, \mathbf{e}_2 \pm \sin\gamma\, \mathbf{e}_3$ (the angle γ is constant). Assuming that the steel fibres of the cord are inextensible, we obtain the equalities

$$\left|\frac{\partial \mathbf{R}_3}{l\partial\xi}\right| = 1, \quad \left|\cos\gamma\frac{\partial \mathbf{R}_3}{l\partial\xi} \pm \sin\gamma\frac{\partial \mathbf{R}_3}{r\partial\varphi}\right| = 1$$

which are equivalent to the equalities

$$\left|l^{-1}\mathbf{R}_3^\circ\right| = 1 \Rightarrow 2lr^{-1}U_2^\circ + \sum_{k=1}^{3} U_k^{\circ 2} = 0; \quad (\cdot)^\circ = \frac{\partial(\cdot)}{\partial\xi}$$

$$\left|r^{-1}\mathbf{R}_3'\right| = 1 \Rightarrow 2(U_1 - U_3') + (U_1 - U_3')^2 + U_2'^2 + (U_1' + U_3)^2 = 0 \tag{1.2}$$

$$\mathbf{R}_3^\circ(\cdot)\mathbf{R}_3' = 0 \Rightarrow U_1^\circ(U_1' + U_3) + (lr^{-1} + U_2^\circ)U_2' - U_3^\circ(1 + U_1 - U_3') = 0; \quad (\cdot)' = \frac{\partial(\cdot)}{\partial\varphi}$$

Henceforth we shall assume that the functions U_k and their derivatives are small; neglecting their squares, we obtain the following equalities from (1.2)

$$U_2^\circ = 0, \quad U_3' = U_1, \quad lr^{-1}U_2' = U_3^\circ \tag{1.3}$$

Denoting the displacements of the centre line l_0 of the tread, corresponding to $\xi = 0$, by $u(\varphi, t)\mathbf{e}_1 + w(\varphi, t)\mathbf{e}_2 - v(\varphi, t)\mathbf{e}_3$, and solving system of equations (1.3), we obtain

$$U_1 = lr^{-1}\xi w'' + u, \quad U_2 = w, \quad U_3 = lr^{-1}\xi w' - v, \quad u = -v' \tag{1.4}$$

Let us consider the case in which the wheel is rolling over the plane OX_1X_2. In the contact region, a part of the tread coincides with the plane OX_1X_2 and is at rest. It follows from conditions (1.2) in the contact region that there are two orthogonal families of straight lines corresponding to constant values of one of the variables φ or ξ in formula (1.1). By formulae (1.4), the shape of the tread in the deformed state is close to that of a ruled surface, and we may assume that the contact region of the tyre with the plane is a rectangle. Let the contact of the tread correspond to values of the angle φ in the range $L_1 = [\varphi_1(t), \varphi_2(t)]$. Then

$$\mathbf{R}_3(\varphi,\xi,t) = -r(\varphi-\pi/2)\mathbf{I}_1 + l\xi\mathbf{I}_2, \quad \varphi \in L_1, \quad |\xi| \leqslant 1 \tag{1.5}$$

In this relation, without loss of generality, we have adopted the convention that the wheel is rolling along the OX_1 axis, its centre line l_0 coinciding with that axis. In view of (1.1), it follows from (1.5) that

$$\begin{aligned}
U_1 &= u + lr^{-1}\xi(\cos\vartheta\sin\beta + \sin\vartheta\sin\varkappa\cos\beta)\\
U_2 &= w + lr^{-1}\xi(\cos\varkappa\cos\beta - 1)\\
U_3 &= -v + lr^{-1}\xi(\sin\vartheta\sin\beta - \cos\vartheta\sin\varkappa\cos\beta)
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
u &= \left(\pi/2 - X_1r^{-1} - \varphi\right)(\cos\beta\cos\vartheta - \sin\beta\sin\varkappa\sin\vartheta) +\\
&+ X_3r^{-1}\sin\vartheta\cos\varkappa - r^{-1}X_2(\cos\vartheta\sin\beta + \sin\vartheta\sin\varkappa\cos\beta) - l\\
w &= -\left(\pi/2 - X_1r^{-1} - \varphi\right)\sin\beta\cos\varkappa - X_3r^{-1}\sin\varkappa - r^{-1}X_2\cos\varkappa\cos\beta\\
v &= -\left(\pi/2 - X_1r^{-1} - \varphi\right)(\cos\beta\sin\vartheta + \sin\beta\sin\varkappa\cos\vartheta) + X_3r^{-1}\cos\vartheta\cos\varkappa +\\
&+ r^{-1}X_2(\sin\vartheta\sin\beta - \cos\vartheta\sin\varkappa\cos\beta); \quad \vartheta = \theta + \varphi
\end{aligned}$$

By (1.4) and (1.6), the relative displacements of the points of the tread in the contact region are determined by X_1, X_2, X_3, β, $\varkappa$, θ as functions of time, but outside the contact region these variables depend on the functions v and w. If we assume $\xi = 0$ in Eqs (1.6), the displacements obtained will correspond to points of the centre line of the tread.

We will define the sidewalls (1) and (2) of the tyre in the deformed state in the form

$$\mathbf{R}_j(\varphi,\psi,t) = \sum_{i=1}^{3} X_i\mathbf{I}_i + \Gamma_3(\beta)\Gamma_1(\varkappa)\Gamma_2(\vartheta)\times\left\{(-1)^j a\mathbf{e}_2 + c\mathbf{e}_1 + b\Gamma_3(\psi)\left[\boldsymbol{\eta}_1 + \sum_{i=1}^{3} V_i\boldsymbol{\eta}_i\right]\right\} \tag{1.7}$$

$$\psi \in I_1 \cup I_2, \quad I_1 = [\psi_1, \psi_2], \quad I_2 = [-\psi_2, -\psi_1], \quad j = 1,2$$

where $V_i(\varphi, \psi, t)$ $(i = 1, 2, 3)$ are the components of the vector of relative displacements of points on the sidewalls in a toroidal system of coordinates $M\eta_1\eta_2\eta_3$ (Fig. 2), and a, b and c are constants. The interval I_j corresponds to the sidewall (j). In a radial tyre, the sidewalls are reinforced by inextensible steel fibres, corresponding in $\mathbf{R}j$ to a constant angle φ, whose curvature will be assumed constant for each fibre under internal pressure in the tyre [6]. These conditions are expressed by the following equalities

$$\left|\frac{\partial\mathbf{R}_j}{b\partial\psi}\right| = 1 \Rightarrow 2\left(V_2 + V_1^{\cdot}\right) + \left(V_2 + V_1^{\cdot}\right)^2 + \left(V_1 - V_2^{\cdot}\right)^2 + V_3^{\cdot 2} = 0$$

$$\left|\frac{\partial^2\mathbf{R}_j}{b\partial\psi^2}\right| = C(\varphi,t) \Rightarrow \left(1 + V_1 - V_1^{\cdot\cdot} + 2V_2^{\cdot}\right)^2 + \left(2V_1^{\cdot} + V_2^{\cdot\cdot} - V_2\right)^2 + V_3^{\cdot\cdot 2} = b^2C^2; \quad (\cdot)^{\cdot} = \frac{\partial(\cdot)}{\partial\psi} \tag{1.8}$$

Assuming that the functions and derivatives in formulae (1.8) are small and neglecting their squares, we obtain a linear system of differential equations

$$V_2 + V_1^{\cdot} = 0, \quad V_1 - V_1^{\cdot\cdot} + 2V_2^{\cdot} = 0$$

whose general solution is

$$V_1 = -c_2 + c_3\sin\psi - c_4\cos\psi, \quad V_2 = c_1 + c_2\psi + c_3\cos\psi + c_4\sin\psi \tag{1.9}$$

where the coefficients $c_k(\varphi, t)$ $(k = 1, \ldots, 4)$ must be determined from the boundary conditions

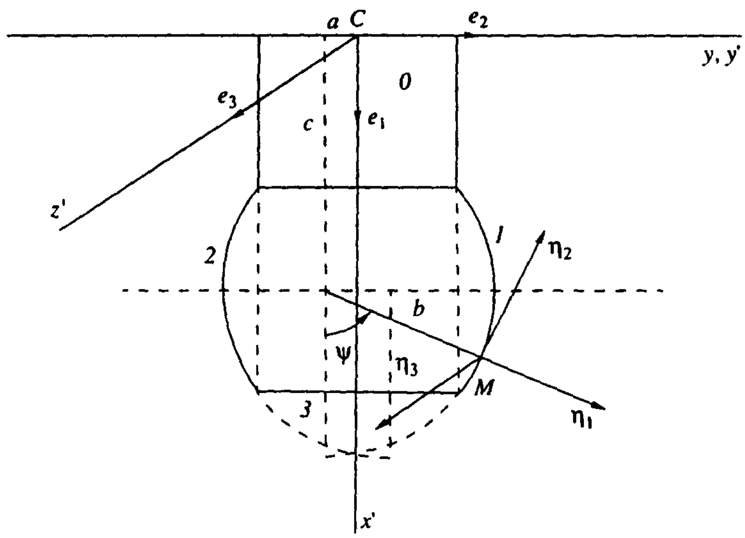

Fig. 2

$$\psi = \pm\psi_2: \quad V_1 = V_2 = 0$$

$$\psi = \pm\psi_1: \quad V_1 = U_{1\pm}\cos\psi_1 \pm U_{2\pm}\sin\psi_1, \quad V_2 = \mp U_{1\pm}\sin\psi_1 + U_{2\pm}\cos\psi_1$$

$$U_{k\pm} = U_k(\pm 1, \varphi, t)$$

As a result, we obtain two systems of linear algebraic equations, whose solution we will express in the form

$$c_k = \frac{r}{b\Delta}\times\begin{cases}\left(-\left[u + lr^{-1}w'' + y_1(1)\right]f_k - \left[w + y_2(1)\right]g_k\right), & \psi \in I_1\\ (-1)^{k+1}\left(\left[u - lr^{-1}w'' + y_1(-1)\right]f_k - \left[w + y_2(-1)\right]g_k\right), & \psi \in I_2\end{cases} \tag{1.10}$$

$$k = 1, \ldots, 4$$

$$f_1 = \psi_2\cos\psi_2 - \psi_1\cos\psi_1 + g_2, \quad f_2 = \cos\psi_1 - \cos\psi_2$$

$$f_3 = \sin(\psi_2 - \psi_1) - (\psi_2 - \psi_1)\cos\psi_1\cos\psi_2$$

$$f_4 = 1 - (\psi_2 - \psi_1)\cos\psi_1\sin\psi_2 - \cos(\psi_2 - \psi_1)$$

$$g_1 = \psi_2\sin\psi_2 - \psi_1\sin\psi_1 - f_2, \quad g_2 = \sin\psi_1 - \sin\psi_2$$

$$g_3 = -1 + \cos(\psi_2 - \psi_1) - (\psi_2 - \psi_1)\sin\psi_1\cos\psi_2$$

$$g_4 = \sin(\psi_2 - \psi_1) - (\psi_2 - \psi_1)\sin\psi_1\sin\psi_2$$

$$\Delta = 2 - 2\cos(\psi_2 - \psi_1) - (\psi_2 - \psi_1)\sin(\psi_2 - \psi_1)$$

where $y_1(\xi)$, $y_2(\xi)$ are additional terms, quadratic in u, v, w and their derivatives, obtained when the functions U_1 and U_2 are determined from conditions (1.2). These terms, to be determined below, turn out to be necessary when calculating the work done by the pressure in the virtual displacements.

The function $V_3(\varphi, \psi, t)$ may be represented approximately by the first two terms in its Taylor series in the neighbourhood of the points $\psi = \pm\psi_2$

$$V_3 = \frac{r}{b}\times\begin{cases}\left(lr^{-1}w' - v\right)\left[1 - (\psi - \psi_1)(\psi_2 - \psi_1)^{-1}\right], & \psi \in I_1\\ \left(-lr^{-1}w' - v\right)\left[1 + (\psi + \psi_1)(\psi_2 - \psi_1)^{-1}\right], & \psi \in I_2\end{cases} \tag{1.11}$$

Let us determine the work done by the pressure to effect the virtual displacements in deformations of the sidewalls and the tread. We have

$$\delta A = \sum_{k=1}^{3} \delta A_k, \quad \delta A_3 = p\int_{-1}^{1}\int_{0}^{2\pi}\left[\mathbf{R}_3' \times \mathbf{R}_3^{\circ}\right]\delta\mathbf{R}_3 d\xi d\varphi,$$
$$\delta A_k = p\int_{l_k}\int_{0}^{2\pi}\left[\mathbf{R}_k' \times \mathbf{R}_k^{\cdot}\right]\delta\mathbf{R}_k d\psi d\varphi, \quad k = 1,2 \tag{1.12}$$

where p is the pressure in the tyre; as shown previously [6], when computing the work of the pressure in (1.12), it may be assumed to be constant, to within terms of the second order of smallness inclusive. For the tread, we obtain from (1.12)

$$\delta A_3 = plr^2\int_{-1}^{1}\int_{0}^{2\pi}\left[\delta U_1 - rl^{-1}U_1^{\circ}\delta U_2 + \left(U_1' + U_3\right)\delta U_3\right]d\xi d\varphi \tag{1.13}$$

To evaluate the integral with respect to ξ, we must use formulae (1.4) and find the quadratic correction to the function U. Denoting the quadratic corrections to the functions defined in (1.4) by y_k and using Eqs (1.2), we obtain

$$y_1 = -\frac{1}{2}w'^2 - \frac{1}{2}\left[lr^{-1}\xi(w''' + w') + u' - v\right]^2 + y_3',$$
$$y_2 = -\frac{l\xi}{2r}\left(w''^2 + w'^2\right), \quad y_3 = \frac{l\xi}{r}(u' - v)w'' \tag{1.14}$$

After integration of the integrand in (1.13) with respect to ξ and a few terms also with respect to φ, we obtain, using (1.4) and (1.14),

$$\delta A_3 = -\delta\Pi_3 = -2plr^2\int_{0}^{2\pi}\left[\frac{l^2}{3r^2}(w''' + w')\delta w''' + (u' - v)\delta u'\right]d\varphi$$
$$\Pi_3 = plr^2\int_{0}^{2\pi}\left[\frac{l^2}{3r^2}\left(w'''^2 - w''^2\right) + v''^2 - v'^2\right]d\varphi \tag{1.15}$$

According to relations (1.12), the work done by the pressure in deforming the sidewalls, to within terms of the second order of smallness inclusive in the functions V_1, V_2, V_3 and their derivatives, is as follows:

$$\delta A_k = pb^3\int_{0}^{2\pi}\int_{l_k}\left[\delta V_1\left(\frac{c}{b} + \cos\psi - V_3' + V_1\cos\psi - V_2\sin\psi\right) - \right.$$
$$\left. -\delta V_2\left(\frac{c}{b} + \cos\psi\right)\left(V_1^{\cdot} - V_2\right) + \delta V_3\left(V_1' + V_3\cos\psi\right)\right]d\varphi d\psi, \quad k = 1,2 \tag{1.16}$$

Let us find the quadratic correction to the function V_1, with an eye to retaining in (1.16) terms of the second order of smallness, inclusive. Replacing V_k in (1.8) by $V_k + z_k$, where z_k are the terms of the second order of smallness, and taking (1.9) into account, we obtain the equalities

$$2\left(z_2 + z_1\right) + \left(c_1 + c_2\psi\right)^2 + V_3{}^2 = 0$$
$$2\left(z_1 - z_1^{\cdot\cdot} + 2z_2^{\cdot}\right) + c_2^2 + \left(c_1 + c_2\psi\right)^2 = 0 \tag{1.17}$$

The quantities c_1 and c_2 are represented by equalities (1.10), assuming that $y_{1,2}(\pm 1)$ are not included, since their contribution to (1.17) consists of terms small to the order of three or more. In addition, the derivation of (1.17) took into account the fact that $1 + c_2 = b^2C^2$ for terms of the first order of smallness. As a result we deduce from (1.17) that

$$z_1'' + z_1 = -V_3'^2 - (c_1 + c_2\psi)^2/2, \quad z_1(\pm\psi_{1,2}) = 0 \tag{1.18}$$

Taking the structure of solution (1.18) and expressions (1.9) and (1.11) into account, we obtain, after rather lengthy computational procedures (integration with respect to ψ, taking the properties of even and odd functions into consideration), an expression for the work done by the pressure to displace the points of the sidewalls

$$\delta A_1 + \delta A_2 = -\int_0^{2\pi}\left[n_0\delta u + \frac{1}{2}n_{01}\delta u^2 + \frac{1}{2}n_{11}\delta u'^2 + m_{21}v'\delta u + m_{12}u'\delta v + \right.$$

$$\left. + \frac{1}{2}m_{22}\delta v^2 + \frac{1}{2}\sum_{j=0}^{3} n_{j3}\delta w^{(j)2}\right]d\varphi, \quad \delta A_1 + \delta A_2 = -\delta\Pi_{1,2} \tag{1.19}$$

The superscript (j) denotes the appropriate derivative of the function with respect to φ, while $\Pi_{1,2}$ is the potential energy functional of the pressure in deformations of the sidewalls. This functional is positive-definite with respect to the variables v and w, if one takes into account that the centre line of the tread is assumed to be inextensible (the second condition in (1.2) with $\xi = 0$),

$$u \approx -v' - (v'' + v)^2/2 - w'^2/2 \tag{1.20}$$

Note that the coefficient n_0 is negative and, as will presently become evident, is equal in absolute value to the tensile strength of the tread under pressure in the tyre.

2. THE EQUATIONS OF MOTION

The kinetic energy of the wheel can be represented in the form

$$2T = m_d\sum_{i=1}^{3}\dot{X}_i^2 + J_{1d}(\dot{\varkappa}^2 + \dot{\beta}^2\cos^2\varkappa) + J_{2d}(\dot{\theta} + \dot{\beta}\sin\varkappa)^2 + \rho r\int_0^{2\pi}\sum_{i=1}^{3}\dot{Z}_i^2 d\varphi \tag{2.1}$$

where m_d and J_{1d}, J_{2d} are the mass and moments of inertia of the disk about the axes Cx, Cy, respectively. The kinetic energy of the tread and the sidewalls is represented in (2.1) by the last term, on the assumption that all the mass of the tyre is uniformly distributed about the centre line l_0, with linear density ρ. The quantities $\dot{Z}_i$ $(i = 1, 2, 3)$ are the projections of the velocity of a point of the centre line of the tread onto the axes of a system of coordinates rotated with respect to the system $OX_1X_2X_3$ through an angle β about the OX_3 axis; they have the form

$$\dot{Z}_1 = \dot{X}_1\cos\beta + \dot{X}_2\sin\beta - r\dot{\beta}[(1+u)\sin\varkappa\sin\vartheta + w\cos\varkappa + v\sin\varkappa\cos\vartheta] -$$

$$- r\dot{\theta}[(1+u)\sin\vartheta + v\cos\vartheta] + r(\dot{u}\cos\vartheta - \dot{v}\sin\vartheta)$$

$$\dot{Z}_2 = -\dot{X}_1\sin\beta + \dot{X}_2\cos\beta + r\dot{\beta}[(1+u)\cos\vartheta - v\sin\vartheta] +$$

$$+ r\dot{\varkappa}[((1+u)\sin\vartheta + v\cos\vartheta)\cos\varkappa - w\sin\varkappa] + r\dot{\theta}[(1+u)\sin\varkappa\cos\vartheta - v\sin\varkappa\sin\vartheta] +$$

$$+ r(\dot{u}\sin\varkappa\sin\vartheta + \dot{w}\cos\varkappa + \dot{v}\sin\varkappa\cos\vartheta) \tag{2.2}$$

$$\dot{Z}_3 = \dot{X}_3 + r\dot{\varkappa}[((1+u)\sin\vartheta + v\cos\vartheta)\sin\varkappa + w\cos\varkappa] -$$

$$- r\dot{\theta}[(1+u)\cos\varkappa\cos\vartheta - v\cos\varkappa\sin\vartheta] -$$

$$- r(\dot{u}\cos\varkappa\sin\vartheta + \dot{v}\cos\varkappa\cos\vartheta - \dot{w}\sin\varkappa); \quad \vartheta = \theta + \varphi$$

The equations of motion and boundary conditions at the as yet unknown contact region are obtained using the Hamilton–Ostrogradskii variational principle. To that end, one needs expressions for the work done by the external forces and torques applied to the disk of the wheel (Fig. 1) in the virtual displacements, namely

$$\delta A_F = F(\beta)\delta X_1 + F(\beta - \pi/2)\delta X_2 - P\delta X_3 + M_1\delta \varkappa + M_2\delta\theta +$$
$$+(M_2 \sin\varkappa + M_3 \cos\varkappa)\delta\beta, \quad F(\beta) = F_1\cos\beta - F_2\sin\beta \tag{2.3}$$

Suppose the wheel is rolling without slip. This means that the velocities of points on the centre line of the tread in the contact region, $L_1 = [\varphi_1(t), \varphi_2(t)$, equal zero, that is, $\dot{Z}_i = 0$ $(i = 1, 2, 3)$. These conditions follow from the holonomic constraints represented by the last three relations in (1.6). The virtual displacements satisfy the equalities $\delta Z_i = 0$ $(i = 1, 2, 3)$, obtained from (2.2) by replacing the time derivatives with the variations of the appropriate functions, and the work of the reactions applied to the centre line is

$$\sum_{i=1}^{3}\delta N_i = \int_{L_1}\sum_{i=1}^{3}\mu_i(\varphi,t)\delta Z_i d\varphi + \sum_{i,k=1}^{2}\mu_{ik}\delta Z_{ik} \tag{2.4}$$

where μ_i, μ_{ik} $(i, k = 2)$ are undetermined Lagrange multipliers – the components of the reaction to the constraints. The subscript k denotes the reaction of the constraints and the virtual displacements at the boundary points of the contact region. Equation (2.4) contains no terms μ_{3k}, since it is assumed that the components of the reactions of the constraints along the OX_3 axis vanish at the boundary points of the contact region.

Yet another constraint on the boundary of the contact region of the tyre, following from (1.1) and (1.4), is that the vector directed along a fibre of the tyre and normal to its centre line is orthogonal to the OX_3 axis, that is,

$$Z_{4k} = \sin\varkappa - w_k''\cos\varkappa\sin\vartheta_k + w_k'\cos\varkappa\cos\vartheta_k = 0 \tag{2.5}$$
$$w_k = w(\varphi_k, t); \quad k = 1, 2$$

The work of the reaction of this constraint (the torque about the OX_1 axis) in virtual displacements will be

$$\delta N_4 = \sum_{k=1}^{2}\mu_{4k}\delta Z_{4k} \tag{2.6}$$

where μ_{4k} and δZ_{4k} are the projection of the torque onto the centre axis of the tread at a boundary point of the contact region and the corresponding virtual displacement.

At the extreme points of the contact region one must consider the torque of the constraints about the OX_3 axis. These reactions are due to the orthogonality of the tread fibres perpendicular to its centre line, the OX_1 axis. Taking formulae (1.1) and (1.4) into account, we express these conditions in the form

$$\frac{\partial \mathbf{R}_3(\varphi_k, 0, t)}{l\partial\xi}\mathbf{I}_1 = 0 \Rightarrow Z_{5k} = -\sin\beta\cos\varkappa + w_k''(\cos\beta\cos\vartheta_k - \sin\beta\sin\varkappa\sin\vartheta_k) +$$
$$+ w_k'(\cos\beta\sin\vartheta_k + \sin\beta\sin\varkappa\cos\vartheta_k) = 0, \quad k = 1, 2, \quad \vartheta_k = \theta + \varphi_k \tag{2.7}$$

We will denote the Lagrange multipliers corresponding to constraints (2.7) (the projections of the torques onto the OX_3 axis at the extreme points of contact) by μ_{5k} $(k = 1, 2)$ and express their work in releasing the constraints in the form

$$\delta N_5 = \sum_{k=1}^{2}\mu_{5k}\delta Z_{5k} \tag{2.8}$$

where δZ_{5k} is the variation of (2.7).

As constraints for the points of the centre line l_0 of the tread outside the contact region we take the condition that it be inextensible (the second relation in (1.2) with $\xi = 0$)

$$2Z_6 = (1 + u + v')^2 + (u' - v)^2 + w'^2 = 1 \tag{2.9}$$

Accordingly, when eliminating these constraints one must take into account the work they perform in virtual displacements

$$\delta N_6 = \int_{L_2} \lambda(\varphi, t)\delta Z_6 d\varphi, \qquad L_2 = [\varphi_2, 2\pi + \varphi_1] \tag{2.10}$$

where λ is an unknown Lagrange multiplier.

Let us write the Hamilton–Ostrogradskii variational principle in the form

$$\int_{t_1}^{t_2} (\delta T + \delta A_F + \sum_{i=1}^{3} \delta A_i + \sum_{i=1}^{6} \delta N_i) dt = 0 \tag{2.11}$$

The corresponding variables in (2.11) are 2π-periodic in φ and the domain of integration $[t_1, t_2] \cup [\varphi_1, \varphi_1 + 2\pi]$ in (2.11) is divided by the curve $\varphi = \varphi_2(t)$ into two parts, $[t_1, t_2] \cup L_1$ and $[t_1, t_2] \cup L_2$, in each of which Green's formula is applicable. The result is the following system of equations

$$-\frac{d}{dt}\nabla_{\dot{x}_1} T + F(\beta) + \int_{L_1} (\mu_1 \cos\beta - \mu_2 \sin\beta) d\varphi + \sum_{k=1}^{2} (\mu_{1k} \cos\beta - \mu_{2k} \sin\beta) = 0$$

$$-\frac{d}{dt}\nabla_{\dot{x}_2} T + F\left(\beta + \frac{\pi}{2}\right) + \int_{L_1} (\mu_1 \sin\beta + \mu_2 \cos\beta) d\varphi + \sum_{k=1}^{2} (\mu_{1k} \sin\beta + \mu_{2k} \cos\beta) = 0$$

$$-\frac{d}{dt}\nabla_{\dot{x}_3} T - P + \int_{L_1} \mu_3 d\varphi = 0$$

$$\nabla_{\varkappa} T - \frac{d}{dt}\nabla_{\dot{\varkappa}} T + M_1 + \int_{L_1} \sum_{i=1}^{3} \mu_i \frac{\partial \dot{Z}_i}{\partial \dot{\varkappa}} d\varphi + \sum_{i=1,2,4,5} \sum_{k=1}^{2} \mu_{ik} \frac{\partial \dot{Z}_{ik}}{\partial \dot{\varkappa}} = 0$$

$$\nabla_{\theta} T - \frac{d}{dt}\nabla_{\dot{\theta}} T + M_2 + \int_{L_1} \sum_{i=1}^{3} \mu_i \frac{\partial \dot{Z}_i}{\partial \dot{\theta}} d\varphi + \sum_{i=1,2,4,5} \sum_{k=1}^{2} \mu_{ik} \frac{\partial \dot{Z}_{ik}}{\partial \dot{\theta}} = 0$$

$$\nabla_{\beta} T - \frac{d}{dt}\nabla_{\dot{\beta}} T + M_2 \sin\varkappa + M_3 \cos\varkappa + \int_{L_1} \sum_{i=1}^{3} \mu_i \frac{\partial \dot{Z}_i}{\partial \dot{\beta}} d\varphi + \sum_{i=1,2,4,5} \sum_{k=1}^{2} \mu_{ik} \frac{\partial \dot{Z}_{ik}}{\partial \dot{\beta}} = 0$$

$$\mu_1 r \cos\vartheta + \mu_2 r \sin\varkappa \sin\vartheta - \mu_3 r \cos\varkappa \sin\vartheta - n_0 - n_{01} u + n_{11} u'' +$$
$$+ m_2(u'' - v') - m_{21} v' = 0, \quad \varphi \in L_1 = [\varphi_1(t), \varphi_2(t)] \tag{2.12}$$

$$\nabla_u T - \frac{d}{dt}\nabla_{\dot{u}} T - n_0 - n_{01} u + n_{11} u'' + m_2(u'' - v') - m_{21} v' + \lambda(1 + u + v') -$$
$$-[\lambda(u' - v)]' = 0, \quad \varphi \in L_2 = [\varphi_2(t), \ \varphi_1(t) + 2\pi], \quad m_2 = 2plr^2$$

$$\rho r^3 [\dot{u}]_k \dot{\varphi}_k - (-1)^k [\lambda(u' - v)]_{l(k)} + (m_2 + n_{11})[u']_k +$$
$$+ r(\mu_{1k} \cos\vartheta_k + \mu_{2k} \sin\varkappa \sin\vartheta_k) = 0, \quad \vartheta_k = \theta + \varphi_k, \quad k = 1, 2$$

$$-\mu_1 r \sin\vartheta + \mu_2 r \sin\varkappa \cos\vartheta - \mu_3 r \cos\varkappa \cos\vartheta - m_{12} u' - m_{22} v = 0, \quad \varphi \in L_1$$

$$\nabla_v T - \frac{d}{dt}\nabla_{\dot{v}} T - m_{12} u' - m_{22} v - [\lambda(1 + u + v')]' - \lambda(u' - v) = 0, \quad \varphi \in L_2$$

$$\rho r^3 [\dot{v}]_k \dot{\varphi}_k - (-1)^k [\lambda(1 + u + v')]_{l(k)} + r(-\mu_{1k} \sin\vartheta_k + \mu_{2k} \sin\varkappa \cos\vartheta_k) = 0; \quad k = 1, 2$$

$$\mu_2 r \cos\varkappa + \mu_3 r \sin\varkappa + \Lambda w = 0, \quad \varphi \in L_1$$

$$(\Lambda w = (m_3 + n_{33}) w^{(6)} + (m_3 - n_{23}) w^{(4)} + n_{13} w'' - n_{03} w = 0), \quad m_3 = 2pl^3/3$$

$$\nabla_w T - \frac{d}{dt}\nabla_{\dot{w}} T + \Lambda w - (\lambda w')' = 0, \quad \varphi \in L_2$$

$$\rho r^3 [\dot{w}]_k \dot{\varphi}_k - (-1)^k [\lambda w']_{l(k)} + (m_3 + n_{33})[w^{(5)}]_k + (m_3 - n_{23})[w''']_k + r\mu_{2k} \cos\varkappa = 0$$

$$\mu_{4k} \cos\varkappa \cos\vartheta_k + \mu_{5k}(\cos\beta \sin\vartheta_k + \sin\beta \sin\varkappa \cos\vartheta_k) - (m_3 + n_{33})[w^{(4)}]_k = 0$$
$$-\mu_{4k} \cos\varkappa \sin\vartheta_k + \mu_{5k}(\cos\beta \cos\vartheta_k - \sin\beta \sin\varkappa \sin\vartheta_k) + (m_3 + n_{33})[w''']_k = 0; \quad k = 1, 2$$

where $[f(\vartheta)]_k = f(\vartheta_k + 0) - f(\vartheta_k - 0)$ is the jump of the function concerned at an end point of the contact region. The subscript $l(k)$ is used to mark value of the appropriate function at $\vartheta_1 - 0$ for $k = 1$ and at $\vartheta_2 + 0$ for $k = 2$.

System (2.12) consists of 12 equations of motion and 10 conditions imposed on the jumps of the functions at the end points of the contact region. Adding the four constraint equations (the last three relations in (1.6) in the contact region condition (2.9) outside it), we obtain a closed system of 26 equations in the unknowns

$$X_1, X_2, X_3, \beta, \varkappa, \theta, \mu_i (i = 1,2,3), \quad \varphi_k, \mu_{ik} (k = 1,2, i = 1,2,4,5), \lambda, u, v, w$$

in the contact region outside it. The determination of the functions u, v and w must also take into consideration the conditions of their continuity at the boundary points of the contact region, which follow from the conditions for the existence of the functionals of the potential energy of the pressure, namely

$$[u]_k = [v]_k = [w]_k = [w']_k = [w'']_k = 0, \qquad k = 1,2 \tag{2.13}$$

3. EQUILIBRIUM OF THE WHEEL AND THE STATIC CHARACTERISTICS OF THE TYRE

If the wheel is equilibrium, all quantities that define its position and the deformation of the tyre, as well as the Lagrange multipliers, the external forces and torques applied to the disk, are independent of the time. Suppose in equilibrium $\theta = 0$, $\varphi \in [\varphi_1, \varphi_2]$, Let us assume that in equilibrium the quantities X_1, X_2, $X_3 - r$, β, $\varkappa$, $\varphi - \pi/2$, $\varphi_2 - \varphi_1$, u, v, w and their derivatives with respect to φ are small. In the contact region, the angle φ is close to $\pi/2$. Using the substitution $\varphi = \pi/2 + \alpha$, $\alpha \in L_1 = [\alpha_1, \alpha_2]$, $\alpha_k = \varphi_k - \pi/2$, we rewrite Eqs (2.12), ignoring quantities of the second and higher order of smallness in these variables. The result is

$$F_1 - F_2\beta + \int_{L_1} \mu_1 d\alpha + \sum_{k=1}^{2} (\mu_{1k} - \mu_{2k}\beta) = 0$$

$$F_1\beta + F_2 + \int_{L_1} \mu_2 d\alpha + \sum_{k=1}^{2} (\mu_{1k}\beta + \mu_{2k}) = 0$$

$$\begin{aligned}
&P = \int_{L_1} \mu_3 d\alpha \\
&M_1 + \int_{L_1} \mu_2 r d\alpha + \sum_{k=1}^{2} (\mu_{2k} r + \mu_{4k}) = 0 \\
&M_2 - \int_{L_1} \mu_1 r d\alpha - \sum_{k=1}^{2} (\mu_{1k} r + \beta\mu_{4k}) = 0 \\
&M_2\varkappa + M_3 + \sum_{k=1}^{2} [\mu_{1k} X_2 - \mu_{2k}(\alpha_k r + X_1) - \mu_{5k}] = 0 \\
&-\mu_1 r\alpha + \mu_2 r\varkappa - \mu_3 r = n_0 - n_{11} - m_2 + (X_3 r^{-1} - 1)(n_{01} - m_{21} + n_{11}), \quad \alpha \in L_1 \\
&-\mu_1 r + \mu_3 r\alpha = m_{12}\alpha + (m_{12} + m_{22}) X_1 r^{-1}, \quad \alpha \in L_1 \\
&\mu_2 r + \mu_3 r\varkappa = -n_{03}(\varkappa + X_2 r^{-1}), \quad \alpha \in L_1 \\
&-n_0 - n_{01} u + n_{11} u'' + m_2(u'' - v') - m_{21} v' + \lambda - [\lambda(u' - v)]' = 0, \quad \alpha \in L_2 \\
&(-1)^k [\lambda(u' - v)]_{l(k)} - (m_2 + n_{11})[u']_k + r(\mu_{1k}\alpha_k - \mu_{2k}\varkappa) = 0 \\
&-m_{12} u' - m_{22} v - \lambda' - \lambda(u' - v) = 0, \quad \alpha \in L_2 \\
&(-1)^k [\lambda]_{l(k)} + r\mu_{1k} = 0, \quad k = 1,2
\end{aligned} \tag{3.1}$$

$$(m_3+n_{33})w^{(6)}+(m_3-n_{23})w^{(4)}+n_{13}w''-n_{03}w-(\lambda w')'=0, \quad \alpha\in L_2$$
$$-(-1)^k[\lambda w']_{l(k)}+(m_3+n_{33})[w^{(5)}]_k+(m_3-n_{23})[w''']_k+r\mu_{2k}=0$$
$$\mu_{4k}\alpha_k-\mu_{5k}+(m_3+n_{33})[w^{(4)}]_k=0$$
$$\mu_{4k}+\mu_{5k}\alpha_k-(m_3+n_{33})[w''']_k=0, \quad k=1,2$$

Relations (3.1) yield μ_1, μ_2 and μ_3 in the contact region L_1 in the form

$$\begin{aligned}\mu_1 r&=(n_{11}+m_2-n_0-m_{12})\alpha-(m_{12}+m_{22})X_1 r^{-1}\\ \mu_2 r&=(n_0-n_{11}-m_2)\varkappa-n_{03}(\varkappa+X_2 r^{-1})\\ \mu_3 r&=-n_0+n_{11}+m_2-(n_{01}-m_{21}+n_{11})(X_3 r^{-1}-1)\end{aligned} \tag{3.2}$$

Putting $\lambda = n_0 + \lambda_1$, where λ_1 is a quantity of the first order of smallness, and using the linearized relation (2.9) $u = -v'$, we can express the equations defining the deformations of the centre line of the tread in the form.

$$\begin{aligned}&(n_{01}+n_0-m_2-m_{21})v'+(n_0-n_{11}-m_2)v'''+\lambda_1=0\\ &(m_{12}+n_0)v''+(n_0-m_{22})v-\lambda_1'=0\\ &(m_3+n_{33})w^{(6)}+(m_3-n_{23})w^{(4)}+(n_{13}-n_0)w''-n_{03}w=0\end{aligned} \tag{3.3}$$

These equations have the following solutions in the domain L_2

$$v(\alpha)=\sum_{i=1}^{4}G_i\exp(p_i\alpha), \quad w(\alpha)=\sum_{j=1}^{6}E_j\exp(q_j\alpha), \quad \alpha\in L_2 \tag{3.4}$$

where p_i ($i = 1,\ldots, 4$), q_j ($j = 1,\ldots, 6$) are the roots of the characteristic equations

$$\begin{aligned}&a_0p^4+a_1p^2+a_2=0\\ &a_0=n_0-m_2-n_{11}, \quad a_1=2n_0+n_{01}-m_2+m_{12}-m_{21}, \quad a_2=n_0-m_{22}\\ &b_0q^6+b_1q^4+b_2q^2+b_3=0\\ &b_0=n_{33}+m_3, \quad b_1=m_3-n_{23}, \quad b_2=n_{13}-n_0, \quad b_3=-n_{03}\end{aligned} \tag{3.5}$$

The roots of characteristic equations (3.5) have the property

$$p_1=-p_3, \quad p_2=-p_4, \quad q_1=-q_4, \quad q_2=-q_5, \quad q_3=-q_6$$

To determine the arbitrary coefficients G_i and E_j in (3.4), we write conditions (2.13) as follows, to within terms of the first order of smallness

$$\begin{aligned}&\sum_{i=1}^{4}G_i\exp(2\pi p_i)=\sum_{i=1}^{4}G_i=X_1r^{-1}\\ &\sum_{i=1}^{4}G_ip_i\exp(2\pi p_i)=\sum_{i=1}^{4}G_ip_i=1-X_3r^{-1}\\ &\sum_{j=1}^{6}E_j\exp(2\pi q_j)=\sum_{j=1}^{6}E_j=-\varkappa-X_2r^{-1}\\ &\sum_{j=1}^{6}E_jq_j\exp(2\pi q_j)=\sum_{j=1}^{6}E_jq_j=\beta\\ &\sum_{j=1}^{6}E_jq_j^2\exp(2\pi q_j)=\sum_{j=1}^{6}E_jq_j^2=0\end{aligned} \tag{3.6}$$

These relations constitute two systems of linear algebraic equations of the fourth and sixth order with constant coefficients in the variables G_i and E_j, respectively. Solving them, we find

$$G_i = \frac{(-1)^i}{2}\exp(-\pi p_i)\frac{\operatorname{sh}(\pi p_1)\operatorname{sh}(\pi p_2)}{\operatorname{sh}(\pi p_i)}[-p_1 p_2 p_i^{-1} B_1 X_1 r^{-1} - B_2(X_3 r^{-1} - 1)], \quad i = 1,\ldots,4 \tag{3.7}$$

$$B_1^{-1} = \operatorname{ch}(\pi p_1) p_2 \operatorname{sh}(\pi p_2) - \operatorname{ch}(\pi p_2) p_1 \operatorname{sh}(\pi p_1)$$

$$B_2^{-1} = \operatorname{sh}(\pi p_1) p_2 \operatorname{ch}(\pi p_2) - \operatorname{sh}(\pi p_2) p_1 \operatorname{ch}(\pi p_1)$$

$$E_j \exp(\pi q_j) = P_{lm}(\varkappa + X_2 r^{-1}) + Q_{lm}\beta \quad (j,l,m)$$

$$E_{j+3}\exp(-\pi q_j) = P_{lm}(\varkappa + X_2 r^{-1}) - Q_{lm}\beta \quad (j,l,m) \tag{3.8}$$

$$P_{lm} = \frac{1}{2\Delta_1} q_l q_m [q_l \operatorname{ch}(\pi q_l)\operatorname{sh}(\pi q_m) - q_m \operatorname{ch}(\pi q_m)\operatorname{sh}(\pi q_l)]$$

$$Q_{lm} = \frac{1}{2\Delta_2}\operatorname{sh}(\pi q_l)\operatorname{sh}(\pi q_m)(q_l^2 - q_m^2)$$

$$\Delta_1 = \begin{vmatrix} \operatorname{ch}(\pi q_1) & \operatorname{ch}(\pi q_2) & \operatorname{ch}(\pi q_3) \\ q_1 \operatorname{sh}(\pi q_1) & q_2 \operatorname{sh}(\pi q_2) & q_3 \operatorname{sh}(\pi q_3) \\ q_1^2 \operatorname{ch}(\pi q_1) & q_2^2 \operatorname{ch}(\pi q_2) & q_3^2 \operatorname{ch}(\pi q_3) \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \operatorname{sh}(\pi q_1) & \operatorname{sh}(\pi q_2) & \operatorname{sh}(\pi q_3) \\ q_1 \operatorname{ch}(\pi q_1) & q_2 \operatorname{ch}(\pi q_2) & q_3 \operatorname{ch}(\pi q_3) \\ q_1^2 \operatorname{sh}(\pi q_1) & q_2^2 \operatorname{sh}(\pi q_2) & q_3^2 \operatorname{sh}(\pi q_3) \end{vmatrix}$$

where the symbol (j, l, m) denotes the set of three equations obtained by cyclic permutation of the indices: $(1, 2, 3) \to (2, 3, 1) \to (3, 1, 2)$.

The conditions imposed on the jumps in (3.1) are

$$\sum_{i=1}^{4} G_i \exp(2\pi p_i \delta_{1k})[(m_2 + n_{11} - n_0)p_i^2 - n_0] = -(m_2 + n_{11})X_1 r^{-1} - (m_2 + n_{11} - n_0)\alpha_k$$

$$\sum_{i=1}^{4} G_i \exp(2\pi p_i \delta_{1k})[(n_0 - n_{11} - m_2)p_i^3 + (n_0 + n_{01} - m_2 - m_{21})] - n_0 = (-1)^k r\mu_{1k}$$

$$\sum_{j=1}^{6} E_j \exp(2\pi q_j \delta_{1k})[(n_{33} + m_3)q_j^5 + (m_3 - n_{23})q_j^3 - n_0 q_j] = (-1)^{k+1} r\mu_{2k} \tag{3.9}$$

$$\sum_{j=1}^{6} E_j \exp(2\pi q_j \delta_{1k})(n_{33} + m_3)q_j^3 = (-1)^k \mu_{4k}$$

$$\sum_{j=1}^{6} E_j \exp(2\pi q_j \delta_{1k})(n_{33} + m_3)q_j^4 = (-1)^k \mu_{5k}, \quad k = 1,2$$

It is now necessary to substitute the expressions (3.7) and (3.8) into conditions (3.9) and find the relation among the forces, torques, displacements of the centre of the disk and its rotations, as well as the size and boundaries of the contact region. As a result we obtain, to within quantities of the first order of smallness inclusive

$$F_1 + \mu_{11} + \mu_{12} = 0, \quad F_2 + \mu_{21} + \mu_{22} = 0, \quad P = r^{-1}(n_{11} + m_2 - n_0)(\alpha_2 - \alpha_1) \tag{3.10}$$

$$M_1 = -\sum_{k=1}^{2}(\mu_{2k} r + \mu_{4k}), \quad M_2 = (\mu_{11} + \mu_{12})r, \quad M_3 = \mu_{51} + \mu_{52}$$

$$\alpha_2 - \alpha_1 = 2(p_1^2 - p_2^2)\operatorname{sh}(\pi p_1)\operatorname{sh}(\pi p_2) B_2 (1 - X_3 r^{-1})$$

$$\alpha_1 + \alpha_2 = 2(p_1 p_2 B_2^{-1} B_1 - 1)X_1 r^{-1}$$

and further

$$\begin{aligned}
&F_1 = 2(n_0 - n_{11} - m_2)(p_1^2 - p_2^2)p_1 p_2 \operatorname{sh}(\pi p_1)\operatorname{sh}(\pi p_2)B_1 X_1 r^{-2}\\
&F_2 = -4r^{-1} \sum_{(j,l,m)} \{q_j \operatorname{sh}(\pi q_j)[(n_{33}+m_3)q_j^4 + (m_3 - n_{22})q_j^2 - n_0]P_{lm}\}(\varkappa + X_2 r^{-1})\\
&P = r^{-1}(n_{11} + m_2 - n_0)(\alpha_2 - \alpha_1)\\
&M_1 = -rF_2 + 4(n_{33}+m_3)\sum_{(j,l,m)}\{q_j^3 \operatorname{sh}(\pi q_j)P_{lm}\}(\varkappa + X_2 r^{-1})\\
&M_2 = -rF_1, \qquad M_3 = -4(n_{33}+m_3)\sum_{(j,l,m)}\{q_j^4 \operatorname{sh}(\pi q_j)Q_{lm}\}\beta
\end{aligned} \tag{3.11}$$

where summation over (j, k, l) means summation over the cyclic permutations of the indices: (1, 2, 3) → (2, 3, 1) → (3, 1, 2).

It follows from relations (3.11) that, in the case of equilibrium of the wheel with the tyre, the force F_1 and torque M_2 are related to one another and depend on the displacement of the centre of the disk with respect to the OX_1 axis, while the force F_2 and torque M_1 are also related and depend on the inclination of the middle plane of the disk and the displacement of its centre along the OX_2 axis. The width of the contact region is proportional to the magnitude of the load P, and its displacement along the OX_1 axis is proportional to the force F_1. We have thus found all the characteristics of the deformed state of the tyre in an equilibrium position, namely, the conditions imposed on the forces and torques applied to the disk of the wheel, and their magnitudes, the contact region of the tyre with the plane and its position, and the shape of the tyre outside the contact region.

4. ROLLING OF A WHEEL WITH CONSTANT VELOCITY

If there is no slip in the contact region of the tyre tread with the plane, the path of the wheel on the contact plane during the rolling may be represented as a straight strip left by the tread of the tyre. The complete system of equations describing these motions is represented by the appropriate relations in Section 2. Among these motions there may be steady rolling motions of the wheel, when the centre of the wheel moves in a straight line parallel to the OX_1 axis at a constant velocity and the angles β and $\varkappa$ are constant (fixed swivel and tilt of the wheel). This motion is described by the relations

$$\begin{aligned}
&\dot{X}_1 = c, \quad X_2 = \text{const}, \quad X_3 = \text{const}, \quad \beta = \text{const}, \quad \varkappa = \text{const}, \quad \dot{\theta} = \Omega\\
&\dot{\varphi}_k(t) = -\Omega, \quad \alpha = \varphi + \Omega t - \pi/2 \quad (u, v, w)(\varphi, t) = (U, V, W)(\alpha)\\
&\mu_i(\varphi, t) = \mu_i(\alpha), \quad i = 1,2,3; \quad \mu_{ik} = \text{const}, \quad \alpha_k = \varphi_k(t) + \Omega t - \pi/2 = \text{const}\\
&i = 1,2,4,5; \quad k = 1,2; \quad \lambda(\varphi, t) = \lambda(\alpha)
\end{aligned} \tag{4.1}$$

The investigation of this steady motion is largely similar to that of the equilibrium of the wheel with the tyre as was done in Section 3. To fix our ideas, let us set

$$\theta = \Omega t, \quad X_1 = ct + \Delta X_1, \quad c = \Omega r \tag{4.2}$$

The last relation in (4.2) follows from the inextensibility of the tread and the condition that the wheel is rolling without slip. The first nine equations of system (3.1) remain unchanged, except for the replacement of X_1 by ΔX_1 (to be done in all the equations of Section 3), while the last eight equations become

$$\begin{aligned}
&g_0(1 + U - U'' + 2V') - n_0 - n_{01}U + n_{11}U'' +\\
&+ m_2(U'' - V') - m_{21}V' + \lambda - [\lambda(U' - V)]' = 0, \quad \alpha \in L_2\\
&g_0[U']_k + (-1)^k[\lambda(U' - V)]_{l(k)} - (m_2 + n_{11})[U']_k + r(\mu_{1k}\alpha_k - \mu_{2k}\varkappa) = 0
\end{aligned}$$

$$g_0(V - V'' - 2U') - m_{12}U' - m_{22}V - \lambda' - \lambda(U' - V) = 0, \quad \alpha \in L_2$$
$$g_0[V']_k + (-1)^k[\lambda]_{l(k)} + r\mu_{1k} = 0, \quad k = 1,2, \quad g_0 = \rho r^3\Omega^2 \tag{4.3}$$
$$-g_0W'' + (m_3 + n_{33})W^{(6)} + (m_3 - n_{23})W^{(4)} + n_{13}W'' - n_{03}W - (\lambda W')' = 0, \quad \alpha \in L_2$$
$$-g_0[W']_k - (-1)^k[\lambda W']_{l(k)} + (m_3 + n_{33})[W^{(5)}]_k + (m_3 - n_{23})[W''']_k + r\mu_{2k} = 0$$
$$\mu_{4k}\alpha_k - \mu_{5k} + (m_3 + n_{33})[W^{(4)}]_k = 0$$
$$\mu_{4k} + \mu_{5k}\alpha_k - (m_3 + n_{33})[W''']_k = 0; \quad k = 1,2$$

Putting $\lambda = n_0 - g_0 + \lambda_1$ in the first, third and fifth equations of system (4.3), we obtain a system of the same form as (3.3). Note that in this case the stress in the tread of the rotating wheel is increased owing to the centrifugal forces added to the internal pressure in the tyre. The solution of the system, assuming that the tread satisfies the linearized inextensibility condition $U = -V'$, turns out to be identical with solution (3.4), with the same values of the roots of characteristic equations (3.5). Equations (3.6) have the same form as before, so that solutions (3.7), (3.8) also remain valid. Conditions (3.9) imposed on the jumps are replaced by the corresponding conditions from (4.3). They may then be expressed in the form which is obtained from (3.9) by replacing X_1 with ΔX_1, adding g_0 to the coefficients of G_i and ΔX_1 in the first equation, adding $-g_0 p_i$ to the coefficient of G_i, adding $(2 - X_3 r^{-1})$ to the left-hand side of the second equation, and, finally, replacing $-n_0 q_j$ with $-(n_0 - g_0)q_j$ in the third equation.

In this steady rolling of the wheel, all the forces and torques applied to the disk are independent of time. The final summary of the relations among the forces, torques and variables defining the motion of the system is represented by the last two relations in (3.10) with X_1 replaced by ΔX_1 in the second relation, as well as relations (3.11), also with X_1 replaced by ΔX_1 and the coefficient n_0 replaced by $n_0 - g_0$ in the formula for the force F_2.

Thus, for the steady rolling motion considered here of a wheel with a reinforced tyre, we have found all the parameters that define the shape of the deformed tyre, the contact region and its position, and the forces and torques applied to the disk of the wheel. Steady motion will exist when all forces and torques are constant and are moreover subject to the modified conditions (3.10) and (3.11). This means that the magnitudes of the forces F_1 and F_2 are proportional to those of the torques M_2 and M_1, respectively. The size of the contact region is proportional to the load P, and its displacement along the OX_1 axis is proportional to the force F_1. The swivel $\varkappa$ of the wheel and its tilt β generate torques M_1 and M_3 applied to the disk of the wheel. This property enables one to eliminate clearances in the suspension of an automobile during its motion and to improve its controllability.

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